A NOTE ON THE CONSTRUCTION OF ASYMMETRICAL ROTATABLE DESIGNS WITH BLOCKS

BY

ANANTA DEB DAS*
Institute of Agricultural Research Statistics
New Delhi

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1. INTRODUCTION

Since the exploration of rotatable designs by Box and Hunter [1] a good number of researchers have worked on the rotatable designs of symmetric type. But there are certain situations in practice, where it is desirable to include different number of levels for different factors. Keeping this end in view, Mehta and Das [3] suggested a method of constructing rotatable designs which are asymmetric in nature.

This note suggests a simple technique of obtaining asymmetrical rotatable designs (ARD) split into blocks which are generated through the operation of orthogonal transformation on the blocks of suitable symmetrical rotatable designs.

2. CONSTRUCTION OF ARD WITH BLOCKS

Let there are 'b' blocks in a symmetrical second order rotatable design (SORD), either central composite or derived from Balanced Incomplete Block Design (BIBD). Let further D be the $(n \times v)$ matrix of the design points of a block of the SORD, where v is the number of factors present in the SORD and n is the number of design points

*Present address—Jute Agriculture Research Institute, Barrackpore West Bengal.

in the particular block. Then the design matrix, D is given by

$$D = \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{v1} \\ x_{12} & x_{22} & \cdots & x_{v2} \\ \vdots & \vdots & & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{vn} \end{bmatrix} \quad \ldots(2.1)$$

where x_{iu} denotes the level of i-th factor in the u-th experimental point (treatment combination) of the block of the SORD, $i=1, 2,\ldots v$ and $u=1, 2, \ldots, n$.

If the blocking arrangement in the symmetrical, SORD is orthogonal then the design points of D given in (2.1) will satisfy the following relations (Box and Hunter [1]) :

$$\left.\begin{array}{lll} (i) & \sum_u^n x_{iu}=0, & (i=1, 2, \ldots v) \\ (ii) & \sum_u^n x_{iu}\, x_{ju}=0, & (i\neq j,\ i, j=1, 2, \ldots, v) \\ (iii) & \sum_u^n x_{iu}^2 = n\lambda_2 & (i=1, 2, \ldots, v) \end{array}\right\} \ldots(2.2)$$

where $\lambda_2=\sum_u^N x_{iu}^2/N$, N being the total number of design points in the SORD.

We, then have,

$$D'D = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{v1} & x_{v2} & \cdots & x_{vn} \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{v1} \\ x_{12} & x_{22} & \cdots & x_{v2} \\ \vdots & \vdots & & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{vn} \end{bmatrix}$$

$$= \begin{bmatrix} n\lambda_2 & 0 & \cdots & 0 \\ 0 & n\lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & n\lambda_2 \end{bmatrix} \quad \left(\begin{array}{c}\text{by the relations } (ii) \\ \text{and } (iii) \text{ of } (2.2)\end{array}\right)$$

i.e. $D'D = n\lambda_2\, I_v$ ...(2.3) where D′ is the transpose of the matrix D and I_v is unit matrix of order v.

Let A be a matrix defined as

$$A = DT \qquad \ldots(2.4)$$

where T is a $(v \times v)$ orthogonal matrix and D is as defined earlier.

Now, if x'_{iu} and t_{ij} denote respectively the elements of the matrices A and T, then (2.4) can be written as

$$\begin{bmatrix} x'_{11} & x'_{21} & \cdots & x'_{v1} \\ x'_{12} & x'_{22} & \cdots & x'_{v2} \\ \vdots & \vdots & & \vdots \\ x'_{1n} & x'_{2n} & \cdots & x'_{vn} \end{bmatrix} = \begin{bmatrix} \sum_i^v x_{i1}t_{1i} & \sum_i^v x_{i1}t_{2i} & \cdots & \sum_i^v x_{i1}t_{vi} \\ \sum_i^v x_{i2}t_{1i} & \sum_i^v x_{i2}t_{2i} & \cdots & \sum_i^v x_{i2}t_{vi} \\ \vdots & \vdots & & \vdots \\ \sum_i^v x_{in}t_{1i} & \sum_i^v x_{in}t_{2i} & \cdots & \sum_i^v x_{in}t_{vi} \end{bmatrix}$$

Thus it follows that any element of the transformed matrix A can be expressed as

$$x'_{ju} = \sum_i^v x_{iu}t_{ji}, \qquad \begin{matrix} (i, j = 1, 2 \ldots, v) \\ (u = 1, 2 \ldots, n) \end{matrix} \qquad \ldots(2.5)$$

Utilizing the above transformation, therefore, one can get the point $(x'_{1u}, x'_{2u}, \ldots, x'_{vu})$ from the design point $(x_{1u}, x_{2u}, \ldots, x_{vu})$. Since x'_{ju} need not be equal to x_{iu}, the transformed levels of any factor may be different from its original levels and the different factors may also have different levels.

Now, since D is the design matrix of a block of symmetrical SORD, the matrix A can also be considered as the design matrix of a block (*i.e.*, the rows of the matrix A can be considered as the design points or treatment combinations of a block) of a new SORD in which the factors may have different number of levels, provided,

(*a*) $A'A$ takes the form as that of $D'D$ shown earlier in (2.3) and

(*b*) the elements of the matrix A satisfy the relation, $\sum_u^n x'_{ju} = 0, \; (j = 1, 2, \ldots, v)$ $\qquad \ldots(2.6)$

We now prove the following :

Leema. The matrix A as defined in (2.4) satisfies the conditions (a) and (b) given in (2.6).

Proof. We need show only the condition (b). since the condition (a) can easily be proved from (2.4) with the help of relation (2.3).

We have from (2.5)

$$x'_{ju}=\sum_{i}^{v} x_{iu}\, t_{ji}, \quad (i, j=1, 2, \ldots, v)\,(u=1, 2, \ldots, n)$$

then
$$\sum_{u}^{n} x'_{ju}=\sum_{u}^{n}\sum_{i}^{v} x_{iu}\, t_{ji}$$

$$=\left(\sum_{u}^{n} x_{iu}\right)\left(\sum_{i}^{v} t_{ji}\right)$$

i.e.,
$$\sum_{u}^{n} x'_{ju}=0, \quad \left(\text{since, } \sum_{u}^{n} x_{iu}=0\right) \qquad \text{.. (2.7)}$$

Hence the lemma.

Thus all the conditions given in (2.6) are satisfied by the elements of the transformed matrix A. Hence the rows of the matrix A can be considered as design points of a block of a new *SORD* in which the factors having different number of levels. In this way, when all 'b' blocks of the symmetrical *SORD* are operated (by post multiplication) with suitably chosen ($v \times v$) orthogonal matrix separately, the resulting product gives rise to 'b' blocks of a new *SORD* which is symmetric in nature. As shown by Mehta and Das[3] we can always find suitable transformation matrix which gives the transformed design to be an asymmetrical rotatable design.

3. ILLUSTRATIVE EXAMPLE

Consider the symmetrical *SORD* in 4 factors having 3 blocks of 9 each obtained from the *BIB* design with parameters $v=4$, $b=6$, $r=3$, $k=2$, $\lambda=1$ (this is a resolvable *BIB* design). This symmetrical *SORD* with blocks has been constructed following the method of Dey and Das[2]. Each factor of the original *SORD* is having 3 levels, *viz.*, (0 and $\pm a$). Operating the following (4×4) orthogonal (transformation) matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{-3}{\sqrt{20}} & \frac{1}{2} & \frac{-1}{\sqrt{20}} \\ \frac{1}{2} & \frac{-1}{\sqrt{20}} & -\frac{1}{2} & \frac{3}{\sqrt{20}} \\ \frac{1}{2} & \frac{1}{\sqrt{20}} & -\frac{1}{2} & \frac{-3}{\sqrt{20}} \\ \frac{1}{2} & \frac{3}{\sqrt{20}} & \frac{1}{2} & \frac{1}{\sqrt{20}} \end{bmatrix}$$

on each block of the original *SORD* separately, we have the blocks of transformed asymmetrical rotatable design as shown below :

Block I

Original blocks				*Transformed blocks*			
a	a	0	0	a	$-4a/x$	0	$2a/x$
a	$-a$	0	0	0	$-2a/x$	a	$-4a/x$
$-a$	$+a$	0	0	0	$2a/x$	$-a$	$4a/x$
$-a$	$-a$	0	0	$-a$	$4a/x$	0	$-2a/x$
0	0	a	a	a	$4a/x$	0	$-2a/x$
0	0	a	$-a$	0	$-2a/x$	$-a$	$-4a/x$
0	0	$-a$	a	0	$2a/x$	a	$4a/x$
0	0	$-a$	$-a$	$-a$	$-4a/x$	0	$2a/x$
0	0	0	0	0	0	0	0

Block II

Original blocks				*Transformed blocks*			
a	0	a	0	a	$-2a/x$	0	$-4a/x$
a	0	$-a$	0	0	$-4a/x$	a	$2a/x$
$-a$	0	a	0	0	$4a/x$	$-a$	$-2a/x$
$-a$	0	$-a$	0	$-a$	$2a/x$	0	$4a/x$
0	a	0	a	a	$2a/x$	0	$4a/x$
0	a	0	$-a$	0	$-4a/x$	$-a$	$2a/x$
0	$-a$	0	a	0	$4a/x$	a	$-2a/x$
0	$-a$	0	$-a$	$-a$	$-2a/x$	0	$-4a/x$
0	0	0	0	0	0	0	0

Block III

Original blocks				*Transformed blocks*			
a	0	0	a	a	0	a	0
a	0	0	$-a$	0	$-6a/x$	0	$-2a/x$
$-a$	0	0	a	0	$6a/x$	0	$-2a/x$
$-a$	0	0	$-a$	$-a$	0	$-a$	0
0	a	a	0	a	0	$-a$	0
0	a	$-a$	0	0	$-2a/x$	0	$6a/x$
0	$-a$	a	0	0	$2a/x$	0	$-6a/x$
0	$-a$	$-a$	0	$-a$	0	a	0
0	0	0	0	0	0	0	0

Here $x = \sqrt{20}$.

We, therefore, see that the transformed design is an asymmetrical *SORD* of the type $3^2 \times 7^2$, whereas the original one is *a* 3^4 symmetrical *SORD*. Again, operating the following (4×4) orthogonal matrix

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

on the same symmetrical *SORD*, we have the transformed asymmetrical *SORD* of the type $5^2 \times 3^2$, the levels of first two factors being $(0, \pm a/\sqrt{2}, \pm \sqrt{2}a)$ and those of last two factors remaining unchanged (*i.e.*, $0, \pm a$).

SUMMARY

A method of constructing Asymmetrical Ratatable Designs (ARD) with blocks has been presented with an illustrative example.

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